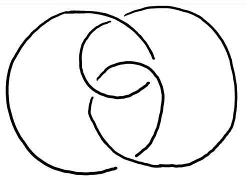
Knot theory, knot practice – Problems 1

LMS Summer School, Bath, July 2025 **Ailsa Keating**

N.B. There are more questions than your group will be about to talk about in an hour! The idea is to have a choice, depending on which topics you found most interesting so far. You're welcome to think about the left-over questions at a later time (but please don't feel obliged to). Questions marked ($\star$) are more exploratory.

1. Convince yourself (by using a string or otherwise) that the Figure-8 knot is equivalent to the following knot diagram:



   Use this to show that the Figure-8 knot is self-mirror. (Such knots are also called 'amphicheiral'.)

2. (a) Draw some examples of torus links $T(p, q)$ for different values of p and q.

   (b) For an arbitrary $n \in \mathbb{Z}$, do $T(1, n)$ and $T(n, 1)$ have simpler names?

   (c) Show that $T(-p, q)$ and $T(p, q)$ are mirror.

   (d) Show that for all $p \in \mathbb{Z}$, we have $T(p, p + 1) = T(p + 1, p)$. [Harder: ($\star$) Show that for all $p, q \in \mathbb{Z}$, we have $T(p, q) = T(q, p)$.]

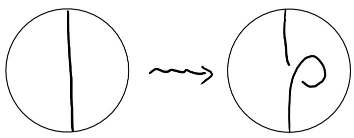
3. Show that there are finitely many links with each crossing number and a fixed number of components.

4. ($\star$) Given a link L, its unlinking number $u(L)$ is defined to be the minimal number of crossing changes required to turn a diagram of L into a diagram of the unlink, where we take the minimum over all diagrams of L. For an arbitrary knot K, show that $u(K)$ satisfies $u(K) \leq \frac{1}{2}c(K)$.

5. Draw some examples of pretzel links $P(p, q, r)$ for small p, q, r. Can you simplify $P(-1, 3, 3)$?

   How many components does $P(p, q, r)$ have depending on the parity of each entry?

6. Using only Reidemeister moves, how can you get the following modification of a knot diagram?



   Hint: You may want to consider the 'Whitney trick' sequence on the back page.

7. ($\star$) Given a link diagram D, show that we can colour the connected components of $\mathbb{R}^2 \backslash D$ in black or white in such a way that no two adjacent components have the same colour. (This is called a chessboard colouring.)

   Use this to construct a surface in $\mathbb{R}^3$ whose boundary is the link represented by D. *[Hint: Consider either all the white components or all the black components.]* Does this surface have to be orientable?

v2, updated 14/07/2025. Comments, corrections or suggestions are all welcome! You can either talk to me in person or write to `amk50@cam.ac.uk`.

Whitney trick: